DOMAIN-VALUED MAXITIVE MAPS
AND THEIR REPRESENTATIONS

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ABSTRACT. The recent extensions of domain theory have proved particularly efficient to study lattice-valued maxitive measures, when the target lattice is continuous. Maxitive measures are defined analogously to classical measures with the supremum operation in place of the addition. Building further on the links between domain theory and idempotent analysis highlighted by Lawson (2004), we introduce the concept of domain-valued *maxitive maps*, which we define as a "point-free" version of maxitive measures. In addition to investigating representations of maxitive maps, we address some extension problems. Our analysis is carried out in the general Z framework of domain theory.

1. INTRODUCTION

Maxitive measures, defined analogously to classical (additive) measures with the supremum operation in place of the addition, have been first introduced by Shilkret [61], and rediscovered and explored by different communities, in particular by mathematicians involved in capacities and large deviations (e.g. Norberg [50], O'Brien and Vervaat [52], Gerritse [25], Puhalskii [57]), idempotent analysis and max-plus algebra (e.g. Maslov [45], Bellalouna [10], Akian et al. [3], Del Moral and Doisy [17], Akian [2]), fuzzy sets (e.g. Zadeh [68], Sugeno and Murofushi [62], Pap [53], de Cooman [15], Nguyen et al. [48], Poncet [54]), or optimisation (e.g. Barron et al. [9], Acerbi et al. [1]), thus many examples of such measures can be found in the literature. For instance, if E is a metric space, the Hausdorff dimension and the Kuratowski measure of non-compacity are maxitive measures on the power set of E. See Falconer [23], Pap [53], Nguyen et al. [48] for further examples.

Many other terms exist to express the same (or a similar) concept, sometimes more popular, yet the term "maxitive", coined by Shilkret, still has our preference because of its anteriority.

Deep connections exist between idempotent analysis and *order theory* or *lattice theory*, since maxitive measures with values in ordered semirings have been considered early [45]. Similar connections have been developed between fuzzy set theory and order theory, when $[0, 1]$-valued possibility measures have been replaced by lattice-valued possibility measures (see Greco [29], Liu and Zhang [38], de Cooman et al. [16], Kramosil [35]). More recently, it appeared that the branch of order theory dealing with *continuous lattices* played a crucial role for the study of lattice-valued maxitive measures. This discovery may be granted to Heckmann and Huth [31, 32], interested in fuzzy set theory, category theory and continuous lattices, and to Akian [2], favouring applications to idempotent analysis and large deviations. Connections between idempotent mathematics and continuous lattices (or domain theory) also arise in the work of Akian and Singer [4], and are surveyed by Lawson [36].

This paper is a contribution to the strengthening of these links. Its goal is to develop the theory of poset-valued maxitive measures with the tools of domain theory. We undertake our analysis in the general Z framework of domain theory (see Bandelt and Erné [6]).

Section 2 gives basic definitions and concepts linked with domains and continuous posets, in the less known general framework of Z-theory. Section 3 introduces the notion of maxitive maps and gives characterizations, while Section 4 focuses on extension theorems. Our maxitive maps take values in a partially ordered set which may not be a complete lattice. When continuity assumptions on this poset are required, we use the tools of Z-theory introduced in Section 2. Section 5 is dedicated to residuated maps and completely maxitive maps, the definitions of which look very close and often coincide. In Section 6 we look at the structure of the set of maxitive maps.

2. A PRIMER ON Z-THEORY FOR CONTINUOUS POSETS AND DOMAINS

A *poset* or *partially ordered set* $(P, \leqslant)$ is a set P equipped with a reflexive, antisymmetric and transitive relation $\leqslant$. Let us denote by <u>Po</u> the category of all posets with order-preserving maps as morphisms. A *subset selection* is a function which assigns to each poset P a certain collection $\mathsf{Z}[P]$ of subsets of P called the Z-*sets* of P. A *subset system* is a subset selection Z such that

   $i)$ at least one $\mathsf{Z}[P]$ has a nonempty member,
   $ii)$ for each order-preserving map $f : P \to Q$, $f(Z) \in \mathsf{Z}[Q]$ for every $Z \in \mathsf{Z}[P]$,

the point $ii)$ meaning that Z is a covariant functor from <u>Po</u> to <u>Set</u> (the category of sets) with $\mathsf{Z}[f]$ defined by $\mathsf{Z}[f](Z) = f(Z)$ if $Z \in \mathsf{Z}[P]$, for every

order-preserving map $f : P \to Q$. This definition was first given by Wright et al. [66]. The suggestion of [66] to apply subset systems to the theory of continuous posets is followed by Nelson [47], Novak [51], Bandelt [5], Bandelt and Erné [6], [7], and this research is carried on by Venugopalan [64], [65], Xu [67], Baranga [8], Menon [46], Shi and Wang [60], Erné [19], [22] among others. Conditions $i)$ and $ii)$ together ensure that each $\mathsf{Z}[P]$ contains all singletons.

The basic example of subset system is the set of directed subsets of P. This subset system is behind the classical theory of continuous posets and domains, see the monograph by Gierz et al. [28]. Here are some further examples:

(1) With $\mathsf{Z}[P]$ the set of filtered subsets of P, one gets the subset system dual to the previous one. It is used for instance by Gerritse [26], Jonasson [34], Akian and Singer [4].

(2) $\mathsf{Z}[P]$ is the set of all singletons of P. With this subset system the way-above relation $y \gg x$ (defined below) reduces to the partial order $y \geqslant x$.

(3) Taking $\mathsf{Z}[P]$ to be the set of all subsets of P works well for investigating completely distributive lattices, see Erné [20]. Completely distributive lattices were first examined by Raney [58], [59].

(4) $\mathsf{Z}[P]$ is the set of chains of P. A series of papers are about that case, see Markowsky and Rosen [43], and Markowsky [40], [41], [42]. With the Hausdorff maximality theorem, relations between directed-complete posets and chain-complete posets and the derived notions of continuity are very strong and are explored by Iwamura [33], Bruns [11], and Markowsky [39].

(5) The case when $\mathsf{Z}[P]$ is the set of finite subsets of P is investigated by Martinez [44], and is linked with the abstract convexity theory developed in van de Vel's monograph [63]. See also Frink [24] and Erné [18].

Rather than Z, we shall often deal with the subset selection F (or $\uparrow\mathsf{Z}$), defined by $\mathsf{F}[P] = \{\uparrow Z : Z \in \mathsf{Z}[P]\}$, where $\uparrow Z$ is the upper subset generated by Z, i.e. $\uparrow Z := \{y \in P : \exists x \in Z, x \leqslant y\}$. The elements of $\mathsf{F}[P]$ are the F-*sets*, or the (Z-)*filters*, of P. F is not a subset system in general, but it satisfies the following conditions:

$\quad i)$ at least one $\mathsf{F}[P]$ has a nonempty member,

$\quad ii')$ for each order-preserving map $f : P \to Q$, $\uparrow f(F) \in \mathsf{F}[Q]$ for every $\quad\quad F \in \mathsf{F}[P]$.

A subset selection F derived from a subset system Z as above will be called a *filter selection*. Note that, like Z, F is functorial, i.e. $\mathsf{F}[g \circ f] = \mathsf{F}[g] \circ \mathsf{F}[f]$

for all order-preserving maps $f : P \to Q$ and $g : Q \to R$, if one naturally defines $\mathsf{F}[f](F) = {\uparrow} f(F)$ for all $F \in \mathsf{F}[P]$.

The first three examples of subset systems given above lead to the following filter selections, respectively:

(1) $\mathsf{F}[P]$ is the set of *filters* (in the sense of [28]) of P,
(2) $\mathsf{F}[P]$ is the set of *principal filters* of P,
(3) $\mathsf{F}[P]$ is the set of *upper sets* of P.

We now introduce the *way-above relation*, which in our context is more relevant than the usual *way-below relation*. Thus, our notions of continuous posets and domains are dual to the traditional definitions. The way-above relation has already been used to study lattice-valued upper semicontinuous functions, see for instance [26] and [34]. We say that $y \in P$ is *way-above* $x \in P$, written $y \gg x$, if, for every F-set F which has an infimum, $x \geqslant \bigwedge F$ implies $y \in F$. We use the notations $\downarrow\!\!\!\downarrow x = \{y \in P : x \gg y\}$, $\uparrow\!\!\!\uparrow x = \{y \in P : y \gg x\}$, and for $A \subset P$, $\downarrow\!\!\!\downarrow A = \{y \in P : \exists x \in A, x \gg y\}$, $\uparrow\!\!\!\uparrow A = \{y \in P : \exists x \in A, y \gg x\}$. The poset P is *continuous* if every element is the F infimum of elements way-above it, i.e. $\uparrow\!\!\!\uparrow x \in \mathsf{F}[P]$ and $x = \bigwedge \uparrow\!\!\!\uparrow x$ for all $x \in P$. P is a *domain* if it is a continuous poset such that every F-set has an infimum in P. For our three examples of subset systems, the notion of continuous posets translates as follows:

(1) P is continuous if and only if it P is (dually) continuous in the usual sense,
(2) every poset is continuous,
(3) P is continuous if and only if it is completely distributive (ou *super-continuous*).

A poset P has the *interpolation property* if, for all $x, y \in P$, if $y \gg x$, there exists some $z \in P$ such that $y \gg z \gg x$. For continuous posets in the classical sense, it is well known that the interpolation property holds, see e.g. [28, Theorem I-1.9]. This is a crucial feature that is behind many important results of the theory. For an arbitrary choice of Z, however, this needs no longer to be true (see the counterexample below). Deriving sufficient conditions on Z to recover the interpolation property is the goal of the following theorem. The subset selection F is *union-complete* if, for every $V \in \mathsf{F}[\mathsf{F}[P]]$ (where $\mathsf{F}[P]$ is considered as a poset ordered by reverse inclusion $\supset$), $\bigcup V \in \mathsf{F}[P]$. As explained in [19], this condition embodies the fact that finite unions of finite sets are finite, $\supset$-filtered unions of filtered sets are filtered, etc. The following theorem restates a result due to [51] and [6] in its dual form. We give the proof here for the sake of completeness.

Theorem 2.1. *If F is a union-complete filter selection, then every continuous poset has the interpolation property.*

Remark 2.2. In the context of Z-theory, many authors (see [51], [6], [64]) call *strongly continuous* a continuous poset with the interpolation property.

Proof. Let P be a continuous poset, and let $x \in P$. We need to show that $F \subset \Uparrow F$, where F denotes the F-set $F = \Uparrow x$. For this purpose we first prove that $\Uparrow F$ is an F-set. Write $\Uparrow F = \bigcup_{y \in F} \Uparrow y = \bigcup V$, where V is the collection of subsets contained in some $\Uparrow y$, $y \in F$. Considering the order-preserving map $f : P \ni y \mapsto \Uparrow y \in \mathsf{F}[P]$ (recall that $\mathsf{F}[P]$ is ordered by reverse inclusion) and using Property ii') above, we have $V = \uparrow f(F) \in \mathsf{F}[\mathsf{F}[P]]$. Since F is union-complete, one has $\Uparrow F = \bigcup V \in \mathsf{F}[P]$. Since P is continuous,

$$x = \bigwedge \Uparrow x = \bigwedge F = \bigwedge_{y \in F} y = \bigwedge_{y \in F} \left(\bigwedge \Uparrow y \right) = \bigwedge \left(\bigcup_{y \in F} \Uparrow y \right) = \bigwedge \Uparrow F.$$

The definition of the way-above relation and the fact that $\Uparrow F \in \mathsf{F}[P]$ give $y \in \Uparrow F = \Uparrow(\Uparrow x)$, for all $y \in \Uparrow x$. Eventually we have shown that P has the interpolation property. $\qquad\square$

All the examples of subsets systems mentioned above are union-complete. It remains an open problem to exhibit a continuous poset with respect to some subset system which does not satisfy the interpolation property.

We should stress the fact that the machinery of category theory is justified as long as relations between posets are examined. If a single poset P is at stake, having just a collection of subsets of P at disposal could be sufficient, as in the works [5], [7], [67] (where the letter $\mathfrak{M}$ is used for the collection of selected subsets). In the present work, however, we hope that the relevance of using functorial (filter) selections will be made clear.

The functors Z and F, which are defined on the category $\underline{\mathsf{Po}}$, could equally be defined on $\underline{\mathsf{Qo}}$ –the category of quasiordered sets with order-preserving maps– or on some subcategory $\underline{\mathsf{K}}$ of $\underline{\mathsf{Qo}}$. Recently some attention has been given to the subset selection defined on the category $\underline{\mathsf{Lat}}$ of lattices and selecting prime (or semiprime) ideals, in order to handle aspects of complete lattices concerning prime and pseudo-prime elements. This led to Zhao's concept of semicontinuous lattices, see [69]. Zhao notices that this theory can not be reduced to the theory of Z-continuous posets, since the choosen subset selection appears not to be a subset system, an apparent drawback being that the derived way-below relation is not included in the partial order of the lattice. Following Zhao's work, Powers and Riedel [56] propose to give up the traditional use of subset systems, and introduce the notion of Z-semicontinuous lattices, with Z a subset selection. Actually, it seems that the categorical aspects could be maintained in this case, if one notices that the selection of prime (or semiprime) ideals of a lattice leads to

a *contravariant* functor $\mathsf{Z} : \underline{\mathsf{Lat}} \to \underline{\mathsf{Set}}$, where for every lattice-morphism $f : L \to L'$ one defines $\mathsf{Z}[f] : \mathsf{Z}[L'] \to \mathsf{Z}[L]$ by $\mathsf{Z}[f](P') = f^{-1}(P')$ for every prime ideal P' of L'.

3. MAXITIVE MAPS

In this section we rely on the concept of maxitive measures to define *maxitive maps*. Such a "point-free" approach may be compared with the works of de Cooman et al. [16] and Comman [14]. These maxitive maps take values in a *poset* (a partially ordered set), which may not be a complete lattice. When continuity assumptions on this poset are required, we use the tools of Z-theory introduced in Section 2.

In this section, F is a union-complete filter selection, and $\overline{E}/E$ and $\overline{L}/L$ are order extensions. An *order extension* Q/P is a pair (P, Q) such that Q is a complete lattice, $P \subset Q$ is equipped with the induced order, and if $A \subset P$ has a supremum (resp. an infimum) in P, then it coincides with its supremum (resp. its infimum) in Q.

Definition 3.1. An *L-valued maxitive map* on E is a map $v : E \to L$ such that

(1)
$$v\left(\bigvee_{j \in J} g_j\right) = \bigvee_{j \in J} v(g_j),$$

for all nonempty finite families $(g_j)_{j \in J}$ of elements of E whose supremum lies in E.

If E is a join-semilattice, a map is maxitive if and only if

(2)
$$v(g \vee g') = v(g) \vee v(g'),$$

for all $g, g' \in E$ with $g \vee g' \in E$, but this equivalence no longer holds for general E, as the following counterexample shows. Let E be the seven-element poset $\{a, b, c, \alpha, \beta, \gamma, z\}$, where

$$a, b \leqslant \alpha, \qquad b, c \leqslant \beta, \qquad c, a \leqslant \gamma, \qquad a, b, c \leqslant z,$$

and no other relation holds. Now take $L = \{0, 1\}$ and $v(z) = 1$, $v(g) = 0$ if $g \in E \setminus \{z\}$. Then v satisfies (2) but is not a maxitive measure since $1 = v(z) = v(a \vee b \vee c) \neq v(a) \vee v(b) \vee v(c) = 0$.

In the particular case where L is the positive cone of a lattice-ordered group R (i.e. $L = \{t \in R : t \geqslant 0\}$) and E is a join-semilattice, the important property that arises should be enhanced. Whenever $v : E \to R$ and $g, g_1, \ldots, g_n \in E$, we classically define (see Choquet [12]) $\Delta_{g_1} \ldots \Delta_{g_n} v(g)$ by iterating the formula $\Delta_{g_1} v(g) = v(g \vee g_1) - v(g)$. Then v is *alternating of infinite order* (or just *alternating*) if

$$(-1)^{n+1} \Delta_{g_1} \ldots \Delta_{g_n} v(g) \geqslant 0,$$

for all $n \in \mathbb{N}^*$, $g, g_1, \ldots, g_n \in E$. Harding et al. [30, Theorem 6.2] enunciate that every $\mathbb{R}_+$-valued maxitive measure is alternating of infinite order. Nguyen et al. [49, Theorem 1] give the same statement based on a combinatorial proof. This is actually true for every L-valued maxitive map, as the following proposition states.

Proposition 3.2. *Assume that E is a join-semilattice and L is the positive cone of a lattice-ordered group. Every L-valued maxitive map on E is alternating.*

Proof. Let v be an L-valued maxitive map on E. Let $g_1, \ldots, g_n \in E$, and define $v_0(g) = -v(g)$, $v_n(g) = (-1)^{n+1}\Delta_{g_n}\ldots\Delta_{g_1}v(g)$. By induction on $n \in \mathbb{N}^*$ the property "$v_n(g \vee g') = v_n(g) \wedge v_n(g')$ and $v_n(g) = 0 \vee (v_{n-1}(g) - v_{n-1}(g_n)) \geqslant 0$, for all $g, g' \in E$" can be shown without difficulty. □

In a poset P, a *lower set* is a subset $A \subset P$ such that $A = {\downarrow}A$, where ${\downarrow}A := \{y \in P : \exists x \in A, y \leqslant x\}$. In this paper, an *ideal* of a poset P should be understood as the empty set or a lower set I such that $\bigvee \Phi \in I$, for every nonempty finite subset Φ of I whose supremum exists in P. Such an ideal is not necessarily directed, so this differs from the standard definition (see for instance Gierz et al. [27]).

The next proposition, which is inspired by Nguyen et al. [49], provides a generic way of constructing a maxitive map from a nondecreasing family of ideals.

Proposition 3.3. *Let $(I_t)_{t \in L}$ be some family of ideals of E such that, for all $g \in E$, $\{t \in L : g \in I_t\}$ is an F-set with infimum. Define $v : E \to L$ by*

$$(3) \qquad v(g) = \bigwedge \{t \in L : g \in I_t\}.$$

If $(I_t)_{t \in L}$ is right-continuous, in the sense that $I_t = \bigcap_{s \gg t} I_s$ for all $t \in L$, then v is maxitive.

Remark 3.4. Assuming that $\{t \in L : g \in I_t\}$ is an F-set for all $g \in E$ makes the family $(I_t)_{t \in L}$ necessarily nondecreasing.

Proof. Let v be given by Equation (3). Obviously, v is order-preserving, so it remains to show that, for all finite family $\{g_j\}_{j \in J}$ of elements of E such that $\bigvee_{j \in J} g_j \in E$, and for every upper bound $m \in L$ of $\{v(g_j)\}_{j \in J}$, we get $m \geqslant v(\bigvee_{j \in J} g_j)$. Let $s \gg m$. One has $g_j \in I_s$ for all $j \in J$, thus $\bigvee_{j \in J} g_j \in I_s$. This implies $\bigvee_{j \in J} g_j \in \bigcap_{s \gg m} I_s = I_m$. Eventually $m \geqslant \bigwedge \left\{r \in L : \bigvee_{j \in J} g_j \in I_r\right\} = v(\bigvee_{j \in J} g_j)$, so v is maxitive. □

When the range L of the maxitive map is continuous, one can remove the assumption of right-continuity of the family of ideals. This leads to the converse statement as follows.

Proposition 3.5. *Assume that L is a continuous poset. A map $v : E \to L$ is maxitive if and only if there is some family $(I_t)_{t \in L}$ of ideals of E such that, for all $g \in E$, $\{t \in L : g \in I_t\}$ is an F-set with infimum and*

$$v(g) = \bigwedge \{t \in L : g \in I_t\}.$$

In this case, (I_t) is right-continuous if and only if $I_t = \{g \in E : t \geqslant v(g)\}$ for all $t \in L$.

Proof. If v is maxitive, simply take $I_t = \{g \in E : t \geqslant v(g)\}$, $t \in L$, which is right-continuous since L is continuous. Conversely, assume that Equation (3) is satisfied. Let $J_t = \bigcap_{s \gg t} I_s$. $(J_t)_{t \in L}$ is a nondecreasing family of ideals of E such that $J_t \supset I_t$ for all $t \in L$. Moreover, $(J_t)_{t \in L}$ is right-continuous thanks to the interpolation property, and by continuity of L one has $v(g) = \bigwedge \{t \in L : g \in J_t\}$. Using Proposition 3.3, v is maxitive.

Assume that (I_t) is right-continuous. The inclusion $I_t \subset \{g \in E : t \geqslant v(g)\}$ is clear. If $t \geqslant v(g)$, we want to show that $g \in I_t$, i.e. $g \in I_s$ for all $s \gg t$. So let $s \gg t \geqslant v(g)$. Equation (3) implies that $g \in I_s$, and the inclusion $I_t \supset \{g \in E : t \geqslant v(g)\}$ is proved. $\qquad\square$

4. EXTENSIONS OF MAXITIVE MAPS

From Proposition 3.5 we can deduce the following corollary, which is useful to extend a maxitive map to the entire complete lattice $\overline{E}$. This improves Heckmann and Huth [32, Proposition 12] and Akian [2, Proposition 3.1], the latter being derived from Maslov [45, Theorem VIII-4.1], but it must be pointed out that all these works only deal with set functions. Henceforth, F is again a union-complete filter selection, and $\overline{E}/E$ and $\overline{L}/L$ are order extensions. The set E^* denotes the collection of all $a \in \overline{E}$ such that $\uparrow a \cap E = \{g \in E : g \geqslant a\}$ is a nonempty F-set in E. This ensures that the set $\uparrow\{v(g) : g \in E, g \geqslant a\}$ is a nonempty F-set (in L) itself. Note that $E^* \supset E$, since $\mathsf{F}[E]$ contains all principal filters.

Corollary 4.1. *Assume that E is a join-semilattice and L is a domain. Let v be an L-valued maxitive map on E. The map $v^* : E^* \to L$ defined by*

$$v^*(a) = \bigwedge_{g \in \uparrow a \cap E} v(g)$$

is maxitive, this is the maximal maxitive map extending v to E^*.

Proof. If v is defined by Equation (3), let $I_t^* = \{a \in E^* : \uparrow a \cap I_t \neq \emptyset\} = \downarrow I_t$. Then $(I_t^*)_{t \in L}$ is a nondecreasing family of ideals of E^*. For all $a \in E^*$, $\{t \in L : a \in I_t^*\} = \bigcup_{g \in \uparrow a \cap E} \{t \in L : g \in I_t\}$ is an F-set in L, since F is union-complete. Then the fact that $v^*(a) = \bigwedge \{t \in L : a \in I_t^*\}$ and Proposition 3.5 show that v^* is maxitive. $\qquad\square$

This corollary also generalises a result due to Kramosil [35, Theorem 15.2], who supposes that L is a complete chain (hence necessarily a continuous complete semilattice).

The following proposition is adapted from Kramosil [35, Theorem 15.1] (the proof is very similar, so is not given here). E_* denotes the collection $\{a \in \overline{E} : \forall g \in E, g \wedge a \in E\}$. If $\overline{E}$ is distributive and E is a join-semilattice, then E_* is a join-semilattice. If E is a meet-semilattice, then $E \subset E_*$ and E_* is a meet-semilattice.

Proposition 4.2. *Assume that $\overline{E}$ is distributive and L is directed-complete. Let v be an L-valued maxitive map on E. The map $v_* : E_* \to L$ defined by*

$$v_*(a) = \bigvee_{g \in \downarrow a \cap E} v(g)$$

is maxitive, and this is the minimal extension of v to E_*.

5. COMPLETELY MAXITIVE MAPS AND RESIDUATED MAPS

In this section, we compare the notions of *residuated map* and *completely maxitive map*. Completely maxitive maps enlarge the usual notion of *completely maxitive measure* or *possibility measure*. Residuated maps are related to adjoint pairs and Galois connections, see Erné [21]. First we need some definitions related to the order extension $\overline{E}/E$. An *ideal* of $\overline{E}/E$ is an ideal of E. A *principal ideal* of $\overline{E}/E$ is an ideal of E such that there is some $a \in \overline{E}$ with $I = \downarrow a \cap E$.

Let $v : E \to L$. v is *completely maxitive* (or is a *sup-map*) if $v(g) = \bigvee_{j \in J} v(g_j)$, for every family $(g_j)_{j \in J}$ of elements of E whose supremum g lies in E. In particular, every completely maxitive map is maxitive. v is *residuated on $\overline{E}/E$* if $I_t := \{g \in E : t \geqslant v(g)\}$ is a principal ideal of $\overline{E}/E$, for all $t \in L$.

In the following lines we give two characterizations of residuated maps.

Proposition 5.1. *Let $v : E \to L$. Then v is residuated on $\overline{E}/E$ if and only if there is some map $w : L \to \overline{E}$ such that*

$$v(g) \leqslant t \Leftrightarrow g \leqslant w(t),$$

for all $g \in E$, $t \in L$.

Proof. To prove the 'only if' part, take $w(t) := \bigvee I_t$. $\qquad\square$

Proposition 5.2. *Let $(I_t)_{t \in L}$ be some family of principal ideals of $\overline{E}/E$ such that, for all $g \in E$, $\{t \in L : g \in I_t\}$ is an F-set with infimum. Define $v : E \to L$ by*

$$v(g) = \bigwedge \{t \in L : g \in I_t\}.$$

Writing $I_t = \downarrow a_t \cap E$, $(I_t)_{t \in L}$ is right-continuous if and only if $(a_t)_{t \in L}$ is right-continuous (or Scott-continuous), in the sense that $a_t = \bigwedge_{s \gg t} a_s$ for all $t \in L$, and in that case v is residuated on $\overline{E}/E$.

A modification of Proposition 3.5 would obviously lead to a similar characterization of residuated maps.

Proof. Similar to the proof of Proposition 3.3. □

Now we are able to draw a link between the notions of residuated map and completely maxitive map. We shall say that a lattice P is *meet-continuous* if

$$x \wedge \bigvee I = \bigvee \downarrow x \cap I,$$

for all ideals I with suprema, and for all $x \in P$.

Theorem 5.3. *Let $v : E \to L$. If v is residuated on $\overline{E}/E$ then v is completely maxitive. Moreover, if $\overline{E}$ is meet-continuous or E is complete, the converse holds.*

Proof. Assume that v is residuated on $\overline{E}/E$. Let $(g_j)_{j \in J}$ be a family of elements of E such that $g := \bigvee_{j \in J} g_j \in E$. Since v is residuated, the set $I := \{h \in E : v(g) \geqslant v(h)\}$ is a principal ideal of $\overline{E}/E$. Hence there is some $a \in \overline{E}$ such that $I = \downarrow a \cap E$. Since $g_j \in I$, $g_j \leqslant a$ for all $j \in J$, which implies $g \in I$. Hence $v(g) \leqslant \bigvee_{j \in J} v(g_j)$. This shows that v is completely maxitive.

Now assume that v is completely maxitive. Let $t \in L$ and $I = \{g \in E : t \geqslant v(g)\}$. Clearly I is an ideal. Let us show that I is principal on $\overline{E}/E$. Let $a = \bigvee I \in \overline{E}$. If $g \in I$ then $g \in \downarrow a \cap E$. Conversely let $g \in \downarrow a \cap E$. Then, if $\overline{E}$ is meet-continuous, $g = g \wedge a = g \wedge \bigvee I = \bigvee \downarrow g \cap I \in E$, and using the fact that v is completely maxitive we get $v(g) = \bigvee_{h \in \downarrow g \cap I} v(h) \leqslant t$, i.e. $g \in I$. If E is complete, then $a \in E$, and thanks to the complete maxitivity of v we get $a \in I$, hence $g \in I$. □

6. STRUCTURE OF THE SET OF MAXITIVE MAPS

Let F be a union-complete filter selection. Basically, one could say that the structure of the set $\mathscr{M}$ of maxitive maps $v : E \to L$ follows the structure of their common range L. We say that a poset P is *complete* if every F-set of P has an infimum in P. Recall that a *domain* is a poset which is both complete and continuous.

Lemma 6.1. *Assume that L is a domain. Then $\mathscr{M}$ is a complete poset, and the infimum of an F-family in $\mathscr{M}$ coincides with the pointwise infimum.*

Proof. Let us consider a family $\mathscr{V}$ of elements of $\mathscr{M}$ such that $\mathscr{V} \in \mathsf{F}[\mathscr{M}]$, and let us show that it has an infimum. For all $v \in \mathscr{V}$, write $v(g) = \bigwedge \{ t \in L : g \in I_t^v \}$, where $I_t^v = \{ g \in E : v(g) \leqslant t \}$. For all $g \in E$, $\uparrow \{ v(g) : v \in \mathscr{V} \} = \uparrow \varphi(\mathscr{V})$, where φ is the order-preserving map $v \mapsto v(g)$, hence $\uparrow \{ v(g) : v \in \mathscr{V} \}$ is an F-set of L. L is complete, so that $\{ v(g) : v \in \mathscr{V} \}$ has an infimum, and it is easily seen that $\bigwedge_{v \in \mathscr{V}} v(g) = \bigwedge \{ t \in L : g \in I_t \}$, where $I_t := \bigcup_{v \in \mathscr{V}} I_t^v$. Now $(I_t)_{t \in L}$ is a nondecreasing family of ideals of E, and using the fact that F is union-complete we get $\{ t \in L : g \in I_t \} \in \mathsf{F}[L]$ for every $g \in E$. Applying Proposition 3.5, one sees that $g \mapsto \bigwedge_{v \in \mathscr{V}} v(g)$ is maxitive, and this is the infimum of $\mathscr{V}$. $\qquad \square$

Lemma 6.2. *Assume that L is a domain with a top $\top$, and let $h \in E$, $s \in L$. Then the map $\langle h, s \rangle : E \to L$ defined by $\langle h, s \rangle (g) = s$ if $h \geqslant g$, $\langle h, s \rangle (g) = \top$ otherwise, is maxitive. Moreover, if $v \in \mathscr{M}$ and $s \gg v(h)$, then $\langle h, s \rangle \gg v$ in $\mathscr{M}$.*

Proof. To show that $\langle h, s \rangle$ is maxitive, use again Proposition 3.5 with $I_t = \downarrow h$ if $t \geqslant s$, $I_t = \emptyset$ otherwise. Assume that $s \gg v(h)$, and let us prove that $\langle h, s \rangle \gg v$ in $\mathscr{M}$. Let $\mathscr{V}$ be an F-set of $\mathscr{M}$ such that $v \geqslant \bigwedge \mathscr{V}$. Then $s \gg v(h) \geqslant \bigwedge_{w \in \mathscr{V}} w(h)$ and $\uparrow \{ w(h) : w \in \mathscr{V} \}$ is an F-set of L (see the proof of Lemma 6.1), so there is some $w_0 \in \mathscr{V}$ such that $s \geqslant w_0(h)$. Now it is easy to see that $\langle h, s \rangle (g) \geqslant w_0(g)$ for all $g \in E$, and we conclude that $\langle h, s \rangle \gg v$. $\qquad \square$

As the reader may have anticipated, we would like to prove that $\mathscr{M}$ is continuous if L is itself continuous. Unfortunately, for $v \in \mathscr{M}$, the family $\{ \langle h, s \rangle : h \in E, s \in L, s \gg v(h) \}$ is not necessarily an F-set. Hence for the remaining part of this paper, we take for F the filter selection that selects filtered upper subsets, and we assume that L is a continuous lattice.

Lemma 6.3. *Assume that L is a continuous lattice. Then for all $v \in \mathscr{M}$,*

$$ v = \bigwedge \{ \langle h, s \rangle : h \in E, s \in L, s \gg v(h) \}. $$

Proof. Thanks to Lemma 6.2 it suffices to prove that $v \geqslant \bigwedge \{ \langle h, s \rangle : h \in E, s \in L, s \gg v(h) \}$. If $t \gg v(g)$, then $t = \langle g, t \rangle (g) \geqslant \bigwedge \{ \langle h, s \rangle (g) : h \in E, s \in L, s \gg v(h) \}$. With the continuity of L we deduce that $v(g) \geqslant \bigwedge \{ \langle h, s \rangle (g) : h \in E, s \in L, s \gg v(h) \}$. $\qquad \square$

Combining the preceding lemmata, we conclude:

Theorem 6.4. *If L is a continuous lattice, then $\mathscr{M}$ is a continuous lattice.*

Corollary 6.5. *Assume that L is a continuous lattice, and let $v, w \in \mathscr{M}$. Then $w \gg v$ if and only if there is exits some $n \in \mathbb{N}$, $h_1, \ldots, h_n \in E$ and $s_1, \ldots, s_n \in L$, such that $s_j \gg v(h_j)$ for all j and $w \geqslant \bigwedge_j \langle h_j, s_j \rangle$.*

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In the last part of this section, we study the properties of $\mathscr{M}$ when L is assumed to be a distributive continuous lattice. First we recall the following theorem.

Theorem 6.6. *Assume that L is a continuous lattice. Then the following conditions are equivalent:*

1. *L is distributive,*
2. *L is a Heyting algebra,*
3. *L is a frame.*

Hence in that case, for all $r, s \in L$, there exists an element of L, denoted by $r \leftarrow s$, such that, for all $t \in L$,

$$s \leqslant r \vee t \Longleftrightarrow (r \leftarrow s) \leqslant t.$$

The notions of Heyting algebra and frame employed here are dual to their traditional definitions. (However in lattices the notion of distributivity is autodual, so no confusion can arise.)

A frame which is a continuous lattice is called a *continuous frame*. Note that, if $r \leqslant s$, then $r \leftarrow s$ is the smallest element of L such that the decomposition $s = r \vee (r \leftarrow s)$ holds. The reader may find a proof of Theorem 6.6 in [28, Theorem I-3.15] based on "spectral arguments". Here we give a self-contained proof.

Proof. $(1 \Rightarrow 2)$. Let $(r \leftarrow s) = \bigwedge F_{r,s}$, where $F_{r,s} := \{t \in L : s \leqslant r \vee t\}$, which is a filtered set thanks to the distributivity of L. Let us show that $s \leqslant r \vee (r \leftarrow s)$. Let m be an upper bound of $\{r, r \leftarrow s\}$, and let $u \gg m$. There is some $t \in L$, $s \leqslant r \vee t$, such that $u \geqslant t$. Hence, $u \geqslant s$, so by continuity of L, $m \geqslant s$, and this proves that $s \leqslant r \vee (r \leftarrow s)$. The other implications of Theorem 6.6 are straightforward. $\square$

Now we reformulate Theorem 6.6 in terms of maxitive maps.

Theorem 6.7. *Assume that L is a continuous frame. Then $\mathscr{M}$ is also a continuous frame. Hence, for all $u, v \in \mathscr{M}$, there exists a maxitive map $u \leftarrow v \in \mathscr{M}$ such that, for all $w \in \mathscr{M}$,*

$$v \leqslant u \vee w \Longleftrightarrow (u \leftarrow v) \leqslant w.$$

Moreover, if $v \geqslant u$, $(u \leftarrow v)$ is the smallest maxitive map on E such that the decomposition $v = u \vee (u \leftarrow v)$ holds.

Proof. It suffices to show that $\mathscr{M}$ is distributive, which is obvious. We still give a "constructive" proof for the existence of $u \leftarrow v$. Let $(u \leftarrow v)(g) = \bigwedge\{t \in L : g \in I_t\}$, where $I_t := \{g \in E : \forall h \leqslant g, v(h) \leqslant u(h) \vee t\}$. $(I_t)_{t \in L}$ is a nondecreasing family of ideals of E, and $\{t \in L : g \in I_t\} = \uparrow(\bigvee_{h \leqslant g} u(h) \leftarrow v(h))$ is a principal filter, for every $g \in E$. From

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Proposition 3.5, we deduce that $u \leftarrow v$ is maxitive. The rest of the proof is analog to the proof of Theorem 6.6. □

7. CONCLUSION AND FUTURE WORK

If α is any cardinal number (and in particular if α is the smallest infinite cardinal $\aleph_0$), all results that are expressed here for *finitely* maxitive maps could be as well transposed to α-maxitive maps (with a straightforward definition), provided that ideals are replaced by their appropriate 'α' counterpart.

A natural continuation of this paper would be to apply our results to maxitive measures. An upcoming paper [55] shall tackle the problem of decomposing maxitive measures into a regular and a singular part.

A second step would be to deal with *maxitive forms*, defined as maxitive maps with an additional homogeneity property, since our framework has the particular benefit to encompass maxitive measures and maxitive forms in the same formalism. This possibility of new fruitful links between domain theory and idempotent analysis, emphasizing the works of Litvinov et al. [37] and Cohen et al. [13], will be examined in future work.

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